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On Functions α-Starlike with Respect to Symmetric Conjugate Points

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We introduce here the notion of functions α-starlike with respect to symmetric conjugate points and derive a convolution theorem in this class. Moreover, a sharp coefficient estimate and a structural formula are given.

1. INTRODUCTION

Let A denote the class of functions $f(z)$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = 1 - f'(0) = 0$. We refer to K, S^*, and C, as usual, as the subclasses of A whose members are convex, starlike (w.r.t. origin) and close-to-convex, respectively (see, e.g., [4]). Let us denote by P the class of functions $p(z)$ which are regular in U and satisfy the conditions $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$ in U.

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We now define two operators D and T as follows:

(1) *The Operator* T. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$, let

$$T(f(z)) = \frac{1}{2}\{f(z) - \overline{f(-\bar{z})}\} = z + \sum_{n=2}^{\infty} \frac{1}{2}\left[a_n - (-1)^n \bar{a}_n\right] z^n.$$

(2) *The Operator* D. For $f \in A$ and n is a positive integer, let

$$D^0 f(z) = f(z), \qquad Df(z) = zf'(z),$$
$$D^{n+1} f(z) = D(D^n f(z)), \qquad n = 1, 2, 3, \ldots .$$

It is easily seen that T and D are well-defined on A and have the following properties:

(I) T and D are linear operators on A;
(II) $DT = TD$;
(III) $TT = T$.

R. Md. El-Ashwah and D. K. Thomas [1] defined the class $S_{\rho c}^*$ of functions starlike with respect to symmetric conjugate points as follows:

DEFINITION 1 [1]. A function $f \in A$ with $(f(z)/z)f'(z) \neq 0$ for all $z \in U$ is said to be starlike with respect to symmetric conjugate points, if it satisfies

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}}\right\} = \operatorname{Re}\left\{\frac{Df(z)}{Tf(z)}\right\} > 0, \qquad z \in U. \tag{1}$$

This class is denoted by $S_{\rho c}^*$, which may be viewed as the class of functions $f(z)$ starlike with respect to the function $\overline{f(-\bar{z})}$ in the sense of K. Sakaguchi [3]. In [1], some properties of this class are studied and a structural formula is established.

The geometric interpretation of the condition (1) is that for every r $(0 < r < 1)$, the point $\overline{f(-\bar{z})}$, $z = re^{i\theta}$ is in the left-hand side of the directional tangent at the point $f(z)$ of the image curve of the circle $\{z : |z| = r\}$ under the function $f(z)$.

In this paper, a new class $S_\rho^*(\alpha)$ of functions α-starlike with respect to symmetric conjugate points is defined and some properties of this class such as an interesting convolution theorem, coefficient estimates, and a structural formula are obtained. In addition, another new class $C_{\rho c}(\alpha)$ of functions α-close-to-convex with respect to symmetric conjugate points is discussed.

DEFINITION 2. A function $f \in A$ with $(f(z)/z)f'(z) \neq 0$ in U is said to be α-starlike with respect to symmetric conjugate points, if it satisfies

$$\operatorname{Re}\left\{ \frac{D\left(\alpha D + (1-\alpha)D^0\right)f(z)}{\left(\alpha D + (1-\alpha)D^0\right)Tf(z)} \right\} > 0, \qquad z \in U \tag{2}$$

for some $\alpha \geq 0$. This class is denoted by $S^*_{\rho c}(\alpha)$. It is clearly seen that $S^*_{\rho c}(0) = S^*_{\rho c}$.

Let us adopt the symbol $D_\alpha = \alpha D + (1-\alpha)D^0$, and $f^*(z) = D_\alpha f(z) = (\alpha D + (1-\alpha)D^0)f(z) = (1-\alpha)f(z) + \alpha z f'(z)$. We see that $f \in S^*_{\rho c}(\alpha)$ is equivalent to $f^*(z) \in S^*_{\rho c}(0)$.

2. THE CLASS $S^*_{\rho c}(\alpha)$ AND HADAMARD PRODUCTS

In order to prove our main results we need the following lemmas:

LEMMA 1 [2]. *If $\varphi \in K$, $g \in S^*$, and $p \in P$, then*

$$\operatorname{Re}\left\{ \frac{(\varphi * pg)(z)}{(\varphi * g)(z)} \right\} = \operatorname{Re}\left\{ \frac{\varphi(z) * p(z)g(z)}{\varphi(z) * g(z)} \right\} > 0, \qquad z \in U,$$

where, $*$ denotes, as usual, the Hadamard product of two functions $f(z)$ and $g(z)$ in A, i.e., if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

LEMMA 2. *If $f \in S^*_{\rho c}$, then $Tf \in S^*$.*

Proof. Setting

$$\frac{Df(z)}{Tf(z)} = \frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} := p(z), \tag{3}$$

we have $p \in P$. We also have

$$\overline{p(-\bar{z})} = \frac{\overline{2(-\bar{z})f'(-\bar{z})}}{f(z) - \overline{f(-\bar{z})}} = \frac{D\overline{(-f(-\bar{z}))}}{Tf(z)}. \tag{4}$$

From (3) and (4), we obtain

$$\frac{DTf(z)}{Tf(z)} = \frac{1}{2}\left\{ \frac{Df(z) + D\overline{(-f(-\bar{z}))}}{Tf(z)} \right\} = \frac{1}{2}\left(p(z) + \overline{p(-\bar{z})} \right),$$

$$z \in U. \tag{5}$$

Since $\operatorname{Re}\{p(z)\} > 0$, for all $z \in U$, and $\operatorname{Re}\{\overline{(p(-\bar{z}))}\} = \operatorname{Re}\{p(-\bar{z})\} > 0$, for all $z \in U$, therefore from (5) we have

$$\operatorname{Re}\left\{\frac{z'(Tf(z))}{Tf(z)}\right\} > 0, \qquad z \in U.$$

This means that $Tf \in S^*$. So the proof of Lemma 2 is complete.

By Lemma 2, we see that if $f \in S^*_{\rho c}$, then f is a close-to-convex function in the sense of Kaplan, i.e., $S^*_{\rho c} \subset C$; therefore f is univalent in U.

From Lemma 2, we immediately have

THEOREM 1. *Let $\alpha \geq 0$. If $f \in S^*_{\rho c}(\alpha)$, then $D_\alpha Tf \in S^*$ and $Tf \in S^*_{\rho c}(\alpha)$.*

Proof. Since $F \in S^*_{\rho c}(\alpha)$ if and only if $f^* = D_\alpha f = (\alpha D + (1-\alpha)D^0)f \in S^*_{\rho c}$, we can see from Lemma 2 that $Tf^*(z) = TD_\alpha f \in S^*$. Further, by using $TD = DT$ we get $TD_\alpha = D_\alpha T$. Therefore, we have $D_\alpha Tf(z) = TD_\alpha f \in S^*$. Moreover, $TT = T$ yields

$$\operatorname{Re}\left\{\frac{D(D_\alpha(Tf(z)))}{D_\alpha(T(Tf(z)))}\right\} = \operatorname{Re}\left\{\frac{D(D_\alpha(Tf(z)))}{D_\alpha(Tf(z))}\right\} > 0, \qquad z \in U.$$

Hence $Tf \in S^*_{\rho c}(\alpha)$. This completes the proof of Theorem 1.

Now, we prove a convolution theorem for the class $S^*_{\rho c}(\alpha)$, $\alpha \geq 0$.

THEOREM 2. *If $f \in S^*_{\rho c}(\alpha)$, $\alpha \geq 0$, and $\varphi \in K$ with real coefficients, then $\varphi * f \in S^*_{\rho c}(\alpha)$.*

Proof. For $f \in S^*_{\rho c}(\alpha)$, we have

$$\frac{DD_\alpha f(z)}{D_\alpha Tf(z)} = p(z) \in P. \tag{6}$$

By Theorem 1, $D_\alpha Tf \in S^*$. Since $\varphi \in K$ with real coefficients, one can easily verify that

$$\begin{aligned} D_\alpha T(\varphi * f)(z) &= \varphi(z) * D_\alpha Tf(z) \\ DD_\alpha(\varphi * f)(z) &= \varphi(z) * DD_\alpha f(z) \end{aligned} \tag{7}$$

hold. From (6) and (7) we obtain

$$\frac{DD_\alpha(\varphi * f)(z)}{D_\alpha T(\varphi * f)(z)} = \frac{\varphi(z) * DD_\alpha f(z)}{\varphi(z) * D_\alpha Tf(z)} = \frac{\varphi(z) * p(z) D_\alpha f(z)}{\varphi(z) * D_\alpha Tf(z)}.$$

By using Lemma 1, we have

$$\operatorname{Re}\left\{\frac{DD_\alpha(\varphi * f)(z)}{D_\alpha T(\varphi * f)(z)}\right\} > 0, \qquad z \in U;$$

hence $\varphi * f \in S^*_{\rho c}(\alpha)$, which completes the proof of Theorem 2.

It is easy to verify that $z/(1-z) \in S^*_{\rho c}(\alpha)$. So, from Theorem 2, we conclude that every convex mapping $\varphi \in K$ with real coefficients belongs to $S^*_{\rho c}$. But $K \not\subset S^*_{\rho c}$, since the function $z/(1 - e^{-(\pi/4)i}z)$ is a convex function but does not belong to $S^*_{\rho c}$.

Corollary 1. *If $f \in S^*_{\rho c}(\alpha)$, $0 \le \alpha \le 1$, then $Tf \in S^*$.*

Proof. The case $\alpha = 0$ has been considered in Lemma 2. We consider now the case $0 < \alpha \le 1$. Let

$$K_\alpha(z) = z + \sum_{n=1}^{\infty} \frac{z^n}{1 + (n-1)\alpha} = \frac{\gamma + 1}{z^\gamma} \int_0^z \frac{t^\gamma}{1-t}\, dt,$$

where $\gamma = 1/\alpha - 1 \ge 0$. It is well known that $K_\alpha(z) \in K$ and has real coefficients. Hence $K_\alpha * f \in S^*_{\rho c}(\alpha)$ by Theorem 2. Theorem 1 yields the result $D_\alpha T(K_\alpha * f(z)) = Tf(z) \in S^*$.

Corollary 2. *If $0 \le \alpha \le 1$, then $S^*_{\rho c}(\alpha) \subset S^*_{\rho c}(0) = S^*_{\rho c}$.*

Proof. If $f \in S^*_{\rho c}(\alpha)$, then $D_\alpha Tf(z) \in S^*$ by Theorem 1. Since

$$\frac{Df(z)}{Tf(z)} = \frac{K_\alpha(z) * \big(DD_\alpha f(z)/D_\alpha Tf(z)\big)\big(D_\alpha Tf(z)\big)}{K_\alpha(z) * D_\alpha Tf(z)},$$

where $K_\alpha(z)$ is defined as in Corollary 1, then by using Lemma 1, we have

$$\operatorname{Re}\left\{\frac{Df(z)}{Tf(z)}\right\} > 0, \qquad z \in U;$$

hence $f \in S^*_{\rho c}$. This completes the proof of Corollary 2.

Theorem 3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{\rho c}(\alpha)$, $\alpha \ge 0$, then*

$$|a_n| \le \frac{n}{1 + (n-1)\alpha}, \qquad n = 2, 3, \dots . \tag{8}$$

Especially, $|a_n| \le 1$ for $f \in S^*_{\rho c}(1)$ and $|a_n| \le n$ for $f \in S^*_{\rho c}(0) = S^*_{\rho c}$. The estimate (8) is sharp and the equality is attained for the function $g_\alpha(z)$ given by

$$g_\alpha(z) = z + \sum_{n=2}^{\infty} \frac{n(i)^{n-1}}{1 + (n-1)\alpha} z^n.$$

Proof. Since $f(z) \in S^*_{\rho c}(\alpha)$ if and only if $f^{\alpha}(z) = D_{\alpha} f(z) = (1-\alpha)f(z) + \alpha z f'(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\alpha]a_n z^n \in S^*_{\rho c} \subset C$, and hence $|(1+(n-1)\alpha)a_n| \leq n$, $n = 2, 3, 4, \ldots$. This implies that the inequality (8) holds.

In order to prove that this estimate is sharp, we only have to show that $g_{\alpha}(z) \in S^*_{\rho c}(\alpha)$, $\alpha \geq 0$. This is equivalent to $D_{\alpha}(g_{\alpha}(z)) \in S^*_{\rho c}$; therefore, it is sufficient to prove that

$$\rho(z) = D_{\alpha}(g_{\alpha}(z)) = z + \sum_{n=2}^{\infty} n(i)^{n-1} z^n = \frac{z}{(1-iz)^2} \in S^*_{\rho c}.$$

Since $\rho(z) \in S^*$ and $T\rho(z) = \rho(z)$, we have

$$\operatorname{Re}\left\{\frac{D\rho(z)}{T\rho(z)}\right\} = \operatorname{Re}\left\{\frac{z\rho'(z)}{\rho(z)}\right\} > 0, \qquad z \in U.$$

This implies that $\rho(z) \in S^*_{\rho c}$. Thus we complete the proof of Theorem 3.

Remark. $g_{\alpha}(z) * z/(1+iz) = z + \sum_{n=2}^{\infty}(n/(1+n-1)\alpha))z^n \notin S^*_{sc}(\alpha)$ $(\alpha \geq 0)$.

THEOREM 4. *A function $f \in S^*_{\rho c}$ if and only if there exist a function $p \in P$ and a function $G \in S^*$ with real coefficients such that G satisfies*

$$\frac{zG'(z)}{G(z)} = \frac{1}{2}\left(p(iz) + \overline{p(i\bar{z})}\right), \tag{9}$$

and

$$f'(z) = i\frac{p(z)G(-iz)}{z}. \tag{10}$$

Proof. Suppose that $f \in S^*_{\rho c}$; then there exists a function $p \in P$ such that

$$f(z) = \frac{p(z)Tf(z)}{z}. \tag{11}$$

By Lemma 2, $Tf(z) \in S^*$, and Lemma 1 yields $G(z) = Tf(z) * (z/(1-iz)) \in S^*$. It is easy to see that all the coefficients of G are real. Moreover, we have

$$Tf(z) = G(z) * \frac{z}{1+iz} = iG(-iz). \tag{12}$$

From (11) and (12), we obtain (10).

We now show that the function G satisfies the condition (9). By using (5), after some computations, we have

$$\begin{aligned}\frac{zG'(z)}{G(z)} = \frac{DG(z)}{G(z)} &= \frac{D\big(Tf(z)*(z/(1-iz))\big)}{Tf(z)*(z/(1-iz))}\\ &= \frac{DTf(z)}{Tf(z)}*\frac{1}{1-iz}\\ &= \frac{1}{2}\big(p(z)+\overline{p(-\bar{z})}\big)*\frac{1}{1-iz}\\ &= \frac{1}{2}\big(p(iz)+\overline{p(i\bar{z})}\big).\end{aligned}$$

Hence the condition (9) holds.

Conversely, for $f \in A$, if there exists a function $p \in P$ and a function $G \in S^*$ with real coefficients such that the conditions (9) and (10) hold, then we can show that $f \in S^*_{\rho c}$.

To do this, first we show that $Tf(z) = iG(-iz)$. From (9) we get

$$G'(-iz) = \frac{i}{2}\big(p(z)+\overline{p(-\bar{z})}\big)\frac{G(-iz)}{z};$$

hence

$$iG(-iz) = \int_0^z G'(-it)\,dt = i\int_0^z \frac{1}{2}\big(p(z)+\overline{p(-\bar{t})}\big)\frac{G(-it)}{t}\,dt. \quad (13)$$

From (10), we have

$$f(z) = i\int_0^z \frac{p(t)G(-it)}{t}\,dt.$$

Therefore after some easy computations, we have

$$\begin{aligned}Tf(z) &= \frac{1}{2}\{f(z)-\overline{f(-\bar{z})}\}\\ &= i\int_0^z \frac{p(z)+\overline{p(-\bar{t})}}{2}\,\frac{G(-it)}{t}\,dt\\ &= iG(-iz) \qquad (\text{from } (13)).\end{aligned} \quad (14)$$

Next, from (10) we have

$$f'(z) = \frac{p(z)Tf(z)}{z}. \tag{15}$$

Since $p \in P$, from (14), we have

$$\operatorname{Re}\left\{\frac{Df(z)}{Tf(z)}\right\} = \operatorname{Re}\left\{\frac{zf(z)}{Tf(z)}\right\} = \operatorname{Re}\{p(z)\} > 0, \qquad z \in U.$$

Therefore, we have $f \in S^*_{\rho c}$, and this completes the proof of Theorem 4.

Since $f \in S^*_{\rho c}(\alpha)$ if and only if $D_\alpha f \in S^*_{\rho c}$, therefore from Theorem 4 we obtain the following result immediately.

THEOREM 5. *A function $f \in S^*_{\rho c}(\alpha)$, $\alpha > 0$ if and only if there exist a function $p \in P$ and $G \in S^*$ with real coefficients and satisfying the condition* (9) *such that*

$$f'(z) = i\frac{\gamma + 1}{z^{\gamma+1}} \int_0^z p(t)G(-it)t^{\gamma-1}\,dt, \tag{16}$$

where $\gamma = 1/\alpha - 1 > -1$.

Proof. Since $f \in S^*_{\rho c}(\alpha)$ if and only if $D_\alpha f \in f \in S^*_{\rho c}$, from (10) in Theorem 4, we have

$$z(D_\alpha f(z))' = ip(z)G(-iz).$$

Now, let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then we have

$$\begin{aligned} z(D_\alpha f(z))' &= z(\alpha Df(z) + (1-\alpha)D^0 f(z))' \\ &= z(\alpha z f'(z) + (1-\alpha)f(z))' \\ &= zf'(z) + \alpha z^2 f''(z) \\ &= z + \sum_{n=2}^{\infty} n[1 + (n-1)\alpha]a_n z^n. \end{aligned}$$

Hence

$$z(D_\alpha f(z))' * k_\alpha(z) = z + \sum_{n=2}^{\infty} n a_n z^n = zf'(z),$$

which implies

$$
\begin{aligned}
zf'(z) &= z\big(D_\alpha f(z)\big)' * \frac{\gamma+1}{z^\gamma}\int_0^z \frac{t^\gamma}{1-t}\,dt \\
&= ip(z)G(-iz) * \frac{\gamma+1}{z^\gamma}\int_0^z \frac{t^\gamma}{1-t}\,dt \\
&= i\frac{\gamma+1}{z^\gamma}\int_0^z p(t)G(-it)t^{\gamma-1}\,dt,
\end{aligned}
$$

which yields (16).

3. THE CLASS $C_{\rho c}(\alpha)$

DEFINITION 3. A function $f \in A$ with $(f(z)/z)f'(z) \neq 0$ in U is said to be α-close-to-convex with respect to symmetric conjugate points if it satisfies

$$\operatorname{Re}\left\{\frac{DD_\alpha f(z)}{D_\alpha T\varphi(z)}\right\} > 0, \qquad z \in U \tag{17}$$

for some $\alpha \geq 0$ and some $\varphi \in S^*_{\rho c}(\alpha)$. This class is denoted by $C_{\rho c}(\alpha)$. Especially, the class $C_{\rho c}(0)$ is denoted by $C_{\rho c}$.

Obviously, $S^*_{\rho c}(\alpha) \subset C_{\rho c}(\alpha)$ for $\alpha \geq 0$. In addition, we see that $f \in C_{\rho c}(\alpha)$ if and only if $D_\alpha f \in C_{\rho c}$.

From Theorem 1 and (17), we can prove

COROLLARY 3. *If $f(z) = z + \sum_{n=2}^\infty a_n z^n \in C_{\rho c}(\alpha)$, $\alpha \geq 0$, then $D_\alpha f \in C$, and hence*

$$|a_n| \leq \frac{n}{1+(n-1)\alpha}, \qquad n = 2, 3, 4, \ldots .$$

This estimate is sharp, the extremal function $g_\alpha(z)$ being given (as in Theorem 3) by

$$g_\alpha(z) = z + \sum_{n=2}^\infty \frac{n(i)^{n-1}}{1+(n-1)\alpha} z^n.$$

By the same way as above, we can prove:

THEOREM 6. $C_{\rho c}(\alpha) \subset C_{\rho c}$, $0 < \alpha \leq 1$.

THEOREM 7. *If $f \in C_{\rho c}(\alpha)$, $\alpha \geq 0$, then $Tf \in C_{\rho c}(\alpha)$ and $D_\alpha Tf \in C$.*

THEOREM 8. *If* $f \in C_{\rho c}(\alpha)$, $\alpha \geq 0$, *and* $k \in K$ *with real coefficients, then* $k * f \in C_{\rho c}(\alpha)$.

By means of Theorem 7 and Theorem 8, we obtain

COROLLARY 4. *Let* $0 \leq \alpha \leq 1$. *If* $f \in C_{\rho c}(\alpha)$, *then* $Tf \in C$.

The proof is the same as in Corollary 1.

Finally, we can similarly prove

THEOREM 9. *A function* $f \in C_{\rho c}(\alpha)$ *if and only if there exist a function* $p \in P$ *and a function* $G \in S^*$ *with real coefficients such that*

$$f'(z) = i\frac{\gamma + 1}{z^{\gamma+1}}\int_0^z p(t)G(-it)t^{\gamma-1}\,dt, \qquad \text{if } \alpha > 0,$$

and

$$f'(z) = i\frac{p(z)G(-iz)}{z}, \qquad \text{if } \alpha = 0.$$

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REFERENCES

1. R. Md. El-Ashwah and D. K. Thomas, Some subclasses of close-to-convex functions, *J. Ramanujan Math. Soc.* **2** (1987), 85–100.
2. St. Ruscheweyh and T. Shiel-Small, Hadamard products of schlict functions and Polya-Schoenberg conjecture, *Comment. Math. Helv.* **48** (1973), 119–135.
3. K. Sakaguchi, A sufficient condition for univalency, *in* ‘‘Topics in Univalent Functions and Its Applications,’’ pp. 8–10, Sîikaisekikenhyûsho Kôkyûroku, Vol. 714, 1990.
4. H. M. Srivastava and S. Owa (Eds.), ‘‘Current Topics in Analytic Function Theory,’’ World Scientific, Singapore, 1992.